Proceedings of the Institute of Mathematics and Mechanics,
National Academy of Sciences of Azerbaijan
Volume 45, Number 1, 2019, Pages 41–51

OSTROWSKI AND DRAGOMIR'S INEQUALITIES IN $\mathcal{A}$-2-INNER PRODUCT SPACES

BEHROOZ MOHEBBI NAJMABADI AND TAYEBE LAL SHATERI

Abstract. In this paper we show that a type of Ostrowski's and Drogomir's inequality are valid in $\mathcal{A}$-2-inner product spaces. We also introduce a class of operators analogous to the finite-rank operators on an $\mathcal{A}$-2-inner product space.

1. Introduction and preliminaries

In 1951 A.M. Ostrowski proved the following interesting theorem [7].

Theorem 1.1. *If x,y,z are real n-tuples such that $x \neq 0$ and*

$$\sum_{i=1}^{n} x_i z_i = 0, \sum_{i=1}^{n} y_i z_i = 1, \tag{1.1}$$

then

$$\sum_{i=1}^{n} z_i^2 \geq \frac{\sum_{i=1}^{n} x_i^2}{\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 - (\sum_{i=1}^{n} x_i y_i)^2}. \tag{1.2}$$

The equality holds in (1.2) if and only if

$$z_k = \frac{y_k \sum_{i=1}^{n} x_i^2 - x_k \sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2 \sum_{i=1}^{n} y_i^2 - (\sum_{i=1}^{n} x_i y_i)^2}, \tag{1.3}$$

for $k \in \{1, 2, ..., n\}$.

When the elements are in the form of L^2-functions, this result was proved by Pearce, Pečarić and Varošanec [8]. H. Šikić and T. Šikić [9] by using of argument based on orthogonal projection in inner product spaces have observed that Ostrowski's inequality as follows:

Theorem 1.2. *Let $(E, \langle ., . \rangle)$ be a real or complex inner product space and $x, y \in E$ two linearly independent vectors. If $z \in E$ is so that*

$$\langle z, x \rangle = 0, \langle z, y \rangle = 1, \tag{1.4}$$

2010 *Mathematics Subject Classification.* 46C50, 26D07.

Key words and phrases. $\mathcal{A}$-2-inner product space, Ostrowski's inequality, Dragomir's inequality.

then

$$\|z\|^2 \geq \frac{\|x\|^2}{\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2}. \tag{1.5}$$

The equality holds if and only if

$$z = \frac{\|x\|^2 y - \langle y, x\rangle x}{\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2}. \tag{1.6}$$

In 2003, S.S. Drogomir by using the elementary topic and the Cauchy-Schwarz inequality in inner product spaces, proved the following form of Ostrowski's Inequality [2].

Theorem 1.3. *Let $(H, \langle ., .\rangle)$ be a real or complex inner product space and $x, y \in H$ two linearly independent vectors. If $z \in H$ is such that $\langle x, z\rangle = 0$. then*

$$|\langle z, y\rangle|^2 \leq \frac{\|z\|^2}{\|x\|^2}(\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2). \tag{1.7}$$

The equality in (1.7) holds if and only if

$$z = \mu(y - \frac{\langle y, x\rangle}{\|x\|^2}x), \tag{1.8}$$

where $\mu \in \mathbb{C}$ is such that $|\mu| = \dfrac{\|x\|\|z\|}{\|x\|^2}(\|x\|^2\|y\|^2 - |\langle x, y\rangle|^2)$.

L.Arambašić and R. Rajić [4] extended Theorem 1.3 to elements of a pre-Hilbert C^*-module as follows.

Theorem 1.4. *Let $\mathcal{A}$ be a C^* -algebra and E be a pre-Hilbert C^*-module over $\mathcal{A}$. Let $x, y \in E$ be two nonzero elements that $\langle x, z\rangle = 0$. Then*

$$|\langle z, y\rangle|^2 \leq \frac{\|z\|^2}{\|x\|^2}(\|x\|^2\langle y, y\rangle - |\langle x, y\rangle|^2). \tag{1.9}$$

The equaity in (1.9) holds if and only if

$$y - \frac{x\langle x, y\rangle}{\|x\|^2} = \frac{z\langle z, y\rangle}{\|z\|^2}. \tag{1.10}$$

In [3], S.S. Dragomir established the following refinement of Buzano's inequality in complex Hilbert space E,

$$\left|\frac{\langle x, z\rangle\langle z, y\rangle}{\|z\|^2} - \frac{\langle x, y\rangle}{\alpha}\right| \leq \frac{\|y\|}{|\alpha|\|z\|}\left(|\alpha - 1|^2|\langle x, z\rangle|^2 + \|z\|^2\|x\|^2 - |\langle x, z\rangle|^2\right), \tag{1.11}$$

where $x, y, z \in E$ and $\alpha, x \neq 0$ and $\alpha \in \mathbb{C}$. The case of equality holds in (1.11) if and only if there exist $\beta \in \mathbb{C}$ such that $\alpha\dfrac{\langle x, z\rangle z}{\|z\|^2} = x + \beta b$.

In this paper we state and prove a type of Ostrowski's and Dragomir's inequality in 2- inner product spaces by allowing the 2-inner product to take values in a C^*-algebra. The concepts of 2-inner products and 2-inner product spaces have been more carefully investigated by many authors in the last four decades. A wide list of references related to this topic can be founed in the book [1]. T. Mahdiabad and A. Nazari [5] and the authors [6] introduced 2-inner product that takes values

in a C^*-algebra. Now, we recall some definitions and basis properties of 2-inner product space over a C^*-algebra from [5, 6].

From now, $\mathcal{A}$ denotes a C^*-algebra.

Definition 1.1. A *pre-Hilbert $\mathcal{A}$-module* is a complex vector space E which is also a right $\mathcal{A}$-module, compatible with the complex algebra structure, equipped with an $\mathcal{A}$-valued inner product $\langle .,. \rangle : E \times E \to \mathcal{A}$ which satisfies the following relations
(I_1) $\langle x, x \rangle \geq 0$ for every $x \in E$,
(I_2) $\langle x, y \rangle = \langle y, x \rangle^*$ for every $x, y \in E$,
(I_3) $\langle x, x \rangle = 0$ if and only if $x = 0$,
(I_4) $\langle xa, yb \rangle = a^* \langle x, y \rangle b$ for every $x, y \in E$ and $a, b \in \mathcal{A}$,
(I_5) $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$ for every $x, y, z \in E$ and $\alpha, \beta \in \mathbb{C}$.

Example 1.1. Let $l^2(\mathcal{A})$ be the set of all sequences $\{a_n\}_{n\in\mathbb{N}}$ of elements of a C^*-algebra $\mathcal{A}$ such that the series $\sum_{n\in\mathbb{N}} a_n a_n^*$ is convergent in $\mathcal{A}$. Then $l^2(\mathcal{A})$ is a Hilbert $\mathcal{A}$-module with respect to the pointwise operations and inner product defined by

$$\langle \{a_n\}_{n\in\mathbb{N}}, \{b_n\}_{n\in\mathbb{N}} \rangle = \sum_{n\in\mathbb{N}} a_n b_n^*.$$

Definition 1.2. Let E be a right $\mathcal{A}$-modulea, an $\mathcal{A}$-combination of $x_1, x_2, ..., x_n$ in E is written as follows

$$\sum_{i=1}^{n} x_i a_i = x_1 a_1 + x_2 a_2 + ... + x_n a_n \quad (a_i \in \mathcal{A}).$$

$x_1, x_2, ..., x_n$ are called $\mathcal{A}$-independent if the equation $x_1a_1 + x_2a_2 + ... + x_na_n = 0$ has exactly one solution, namely $a_1 = a_2 = ... = a_n = 0$, otherwise, we say that $x_1, x_2, ..., x_n$ are $\mathcal{A}$-dependent.
The maximum number of elements in E that are $\mathcal{A}$-independent, is called $\mathcal{A}$-rank of E.

Definition 1.3. Let $\mathcal{A}$ be a C*-algebra and E be a linear space by $\mathcal{A}$-rank greater than 1, which is also a right $\mathcal{A}$-module. We define a function $\langle .,.|. \rangle : E \times E \times E \to \mathcal{A}$ satisfies the following properties
(T_1) $\langle x, x|y \rangle = 0$, If and only if $x = ya$ for $a \in \mathcal{A}$,
(T_2) $\langle x, x|y \rangle \geq 0$ for all $x, y \in E$,
(T_3) $\langle x, x|y \rangle = \langle y, y|x \rangle$ for all $x, y \in E$,
(T_4) $\langle x, y|z \rangle = \langle y, x|z \rangle^*$ for all $x, y, z \in E$,
(T_5) $\langle xa, yb|z \rangle = a^* \langle x, y|z \rangle b$ for all $x, y, z \in E$ and $a, b \in \mathcal{A}$,
(T_6) $\langle \alpha x, y|z \rangle = \overline{\alpha} \langle x, y|z \rangle$ for all$x, y \in E$ and $\alpha \in \mathbb{C}$,
(T_7) $\langle x + y, z|w \rangle = \langle x, z|w \rangle + \langle y, z|w \rangle$ for all $x, y, z, w \in E$.
Then the function $\langle .,.|. \rangle$ is called an $\mathcal{A}$-2- inner product and $(E, \langle .,.|. \rangle)$ is called an $\mathcal{A}$-2-inner product space.

Definition 1.4. [5] Let E be a real vector space that $\mathcal{A}$-rank is greater than 1 and
$p : E \times E \to \mathbb{R}$ be a function such that
(1) $p(x, y) = 0$ if and only if $x, y \in E$ are linearly $\mathcal{A}$ - dependent,
(2) $p(x, y) = p(y, x)$ for every $x, y \in E$,

(3) $p(\alpha x, y) = |\alpha| p(x, y)$, for every $x, y \in E$ and for every $\alpha \in \mathbb{C}$,
(4) $p(x + y, z) \leq p(x, z) + p(y, z)$, for every $x, y, z \in E$.
(5) $P(xa, y) \leq ||a|| p(x, y)$, for every $x, y \in E$ and $a \in \mathcal{A}$.
The function p is called an $\mathcal{A}$-2-norm.

Definition 1.5. Let $(E, \langle ., .|.\rangle)$ be an $\mathcal{A}$-2- inner product space, we define $|., .| : E \times E \to \mathcal{A}$ by $(x, y) \mapsto \langle x, x|y\rangle^{1/2}$, then $|., .|$ is called a 2-$\mathcal{A}$-valued norm and $\|x, y\| = \|\langle x, x|y\rangle\|^{1/2}$ is $\mathcal{A}$-2-norm.

We say that two elements x, y of an $\mathcal{A}$-2-inner product space E are w-orthogonal for $w \in E$, if $\langle x, y|w\rangle = 0$.

2. Ostrowski's Inequality in $\mathcal{A}$-2-inner product space

In this section we give a type of Ostrowski's inequality in 2-$\mathcal{A}$-inner product spaces. In the following proposition, we have two version of the Cauchy-Schwarz inequality.

Proposition 2.1. [5]. *Let $(E, \langle ., .|.\rangle)$ be an $\mathcal{A}$-2-inner product space on a C^*-algebra $\mathcal{A}$. Then for $x, y, z \in E$ the following inequalities hold*

(1) $|\langle x, y|z\rangle|^2 = \langle x, y|z\rangle^* \langle x, y|z\rangle \leq \|\langle x, x|z\rangle\| \langle y, y|z\rangle$.
(2) $\|\langle x, y|z\rangle\|^2 \leq \|\langle x, x|z\rangle\| \|\langle y, y|z\rangle\|$.

In the following lemma, we present the necessary and sufficient condition for the equality of Cauchy Schwarz in the previous proposition.

Lemma 2.1. *Let $\mathcal{A}$ be a C^*-algebra and $(X, \langle ., .|.\rangle)$ be an $\mathcal{A}$-2-inner product space. Then for $x, y, z \in X$ $|\langle x, y|z\rangle|^2 = \|x, z\|^2 |y, z|^2$ if and only if there exists $a \in \mathcal{A}$ such that $y = \frac{1}{\|x,z\|^2} x\langle x, y \mid z\rangle + za$.*

Proof. We may assume that $\|x, z\| = 1$. First let us $y = x\langle x, y \mid z\rangle + za$. Then we have $\langle y, x \mid z\rangle\langle x, y \mid z\rangle$=$\langle y, x \mid z\rangle\langle x, x\langle x, y \mid z\rangle + za \mid z\rangle$=$\langle y, x \mid z\rangle\langle x, x \mid z\rangle\langle x, y \mid z\rangle$, which implies that

$$\begin{aligned} 0 &= \langle x\langle x, y \mid z\rangle + za - y, x\langle x, y \mid z\rangle + za - y \mid z\rangle \\ &= \langle y, x \mid z\rangle\langle x, x \mid z\rangle\langle x, y \mid z\rangle - \langle y, x \mid z\rangle\langle x, y \mid z\rangle + \langle y, y \mid z\rangle. \end{aligned}$$

Hence

$$\langle y, x \mid z\rangle\langle x, y \mid z\rangle = \langle y, y \mid z\rangle.$$

Conversely suppose that $\langle y, x \mid z\rangle\langle x, y \mid z\rangle = \langle y, y \mid z\rangle$. Since

$$\langle y, x \mid z\rangle\langle x, x \mid z\rangle\langle x, y \mid z\rangle \leq \|\langle x, x \mid z\rangle\| \langle y, x \mid z\rangle\langle x, y \mid z\rangle$$

we have

$$\begin{aligned} 0 &\leq \langle x\langle x, y \mid z\rangle - y, x\langle x, y \mid z\rangle - y \mid z\rangle \\ &= \langle y, x \mid z\rangle\langle x, x \mid z\rangle\langle x, y \mid z\rangle - \langle y, x \mid z\rangle\langle x, y \mid z\rangle - \langle y, x \mid z\rangle\langle x, y \mid z\rangle + \langle y, y \mid z\rangle \\ &\leq \langle y, x \mid z\rangle\langle x, y \mid z\rangle - \langle y, x \mid z\rangle\langle x, y \mid z\rangle = 0. \end{aligned}$$

Thus, there exists $a \in \mathcal{A}$ such that $y = x\langle x, y \mid z\rangle + za$. □

Now we state and prove a type of Ostrowski's inequality, in an $\mathcal{A}$-2-inner product space.

Theorem 2.1. *Let $\mathcal{A}$ be a C^*-algebra and E be an $\mathcal{A}$-2-inner product space. Let $x, y, z, w \in E$, $x \neq 0, z \neq 0$ be such that $\langle x, z|w\rangle = 0$, then*

$$|\langle z, y|w\rangle|^2 \leq \frac{\|z, w\|^2}{\|x, w\|^2}(\|x, w\|^2|y, w|^2 - |\langle x, y|w\rangle|^2). \tag{2.1}$$

The equality holds if and only if there exists $a \in \mathcal{A}$ such that

$$y - \frac{x\langle y, x|w\rangle}{\|x, w\|^2} = \frac{z\langle y, z|w\rangle}{\|z, w\|^2} + wa.$$

Proof. Without loss of generality, we can assume that $\|z, w\| = \|x, w\| = 1$. Put $\mu = \ y- \ x \ \langle x, y|w\rangle$, then

$$\langle \mu, z|w\rangle = \langle y - x\langle x, y|w\rangle, z|w\rangle = \langle y, z|w\rangle - \langle y, x|w\rangle\langle x, z|w\rangle = \langle y, z|w\rangle. \tag{2.2}$$

By using part (i) of Proposition 2.1, we get

$$\langle y, z|w\rangle\langle z, y|w\rangle = \langle \mu, z|w\rangle\langle z, \mu|w\rangle \leq \|z, w\|^2\langle \mu, \mu|w\rangle. \tag{2.3}$$

Since

$$\langle y, x|w\rangle\langle x, x|w\rangle\langle x, y|w\rangle \leq \|\langle x, x|w\rangle\|\langle y, x|w\rangle\langle x, y|w\rangle = \langle y, x|w\rangle\langle x, y|w\rangle,$$

we obtain

$$\begin{aligned}\langle \mu, \mu|w\rangle &= \langle y - x\langle x, y|w\rangle, y - x\langle x, y|w\rangle \mid w\rangle \\ &= \langle y, y|w\rangle - \langle y, x|w\rangle\langle x, y|w\rangle \\ &- \langle y, x|w\rangle\langle x, y|w\rangle + \langle y, x|w\rangle\langle x, x|w\rangle\langle x, y|w\rangle \\ &\leq \langle y, y|w\rangle - \langle y, x|w\rangle\langle x, y|w\rangle. \end{aligned} \tag{2.4}$$

From (2.3) and (2.4), we deduce that

$$|\langle z, y|w\rangle|^2 \leq \langle \mu, \mu|w\rangle \leq \langle y, y|w\rangle - |\langle x, y|w\rangle|^2,$$

which proves (2.1). The equality holds if and only if the following conditions hold:
$(i)\langle \mu, z \mid w\rangle\langle z, \mu mid w\rangle = \langle \mu, \mu \mid w\rangle$.
$(ii)\langle y, x \mid w\rangle\langle x, x \mid w\rangle\langle x, y \mid w\rangle = \langle y, x \mid w\rangle\langle x, y \mid w\rangle$.
By Lemma 2.1 and (2.2), for some $a \in \mathcal{A}$ the condition (i) is equivalent to $\mu = \dfrac{z\langle z, \mu \mid w\rangle}{\|z, w\|^2} + wa = z\langle z, \mu \mid w\rangle + wa = z\langle z, y \mid w\rangle + wa$, that is

$$y - x\langle x, y \mid w\rangle = z\langle z, y \mid w\rangle + wa.$$

Since $\langle x, z \mid w\rangle = 0$, we have

$$\begin{aligned}\langle y, x \mid w\rangle\langle x, y \mid w\rangle &= \langle y, x \mid w\rangle \,\langle x, x\langle x, y \mid w\rangle + z\langle z, y \mid w\rangle + wa \mid w\rangle \\ &= \langle y, x \mid w\rangle\langle x, x \mid w\rangle\langle x, y \mid w\rangle.\end{aligned}$$

This completes the proof. □

Corollary 2.1. *Let $\mathcal{A}$ be a C^*-algebra and E be an $\mathcal{A}$-2-inner product space. Let $x, z, w \in E$ be such that $\|x, w\| = \|z, w\| = 1$ and $\langle z, z \mid w\rangle$ or $\langle x, x \mid w\rangle$ is a projection, Then $\langle x, z \mid w\rangle = 0$.*

Proof. If $\langle x, x \mid w\rangle$ is a projection, from Theorem 2.1 by putting $y = x$, we have

$$|\langle z, x|w\rangle|^2 \leq |x, w|^2 - |\langle x, x|w\rangle|^2 = \langle x, x \mid w\rangle - \langle x, x \mid w\rangle^2 = 0,$$

so $\langle x, z \mid w\rangle = 0$. Similarly if $\langle z, z \mid w\rangle$ is a projection, it is enough to put $y = z$ in (2.1) to conclude $\langle x, z \mid w\rangle = 0$. □

In the following, we introduce a class of operators on an $\mathcal{A}$-2-inner product space.

Definition 2.1. Let $\mathcal{A}$ be a C^*-algebra and E be an $\mathcal{A}$-2-inner product space. Recall that a map $T : E \to E$ is $\mathcal{A}$- *linear* if $T(xa) = T(x)a$, for all $x \in E, a \in \mathcal{A}$. A bounded linear map T, for $w \in E$ is called $w-$*positive*, if $\langle Tx, x \mid w\rangle \geq 0$ for all $x \in E$, $T \geq S$ if and only if $T - S \geq 0$ be w-positive for any $w \in E$ and define $\|\!|T\|\!| = sup\{\|\langle Tx, x \mid w\rangle\| : \|x, w\| \leq 1, x, w \in E\}$.

We get the following lemma, trivially.

Lemma 2.2. *If T is an $\mathcal{A}$- linear map on an $\mathcal{A}$-2-inner product space E, then*

$$\begin{aligned}\langle Tx, y \mid w\rangle = &\frac{1}{4}\left(\langle T(x+y), x+y \mid w\rangle - \langle T(x-y), x-y \mid w\rangle\right)\\ &-\frac{i}{4}\left(\langle T(x+iy), x+iy \mid w\rangle - \langle T(x-iy), x-iy \mid w\rangle\right),\end{aligned} \tag{2.5}$$

holds for any $x, y \in E$ and the following conditions are equivalent:
(i) $\|\!|T\|\!| = 0$,
(ii) $\langle Tx, x \mid w\rangle = 0$,
(iii) $\langle Tx, y \mid w\rangle = 0$.

Proposition 2.2. *For $x, y, w \in E$, we define $\mathcal{A}$-linear operator $\Theta_{x,y,w} : E \to E$ by $\Theta_{x,y,w}(z) = x\langle y, z \mid w\rangle$. Then*
(i) $\Theta_{x,y,w}$ *is an $\mathcal{A}$-linear map,*
(ii) $\|\!|\Theta_{x,y,w}\|\!| \leq \|x, w\|\|y, w\|$ *for all $x, y, w \in E$,*
(iii) $\Theta_{x,y,w}$ *is w-positive,*
(iv) $\Theta_{x,y,w}\Theta_{x_1,y_1,w_1} = \Theta_{x\langle y,x_1|w\rangle,y_1,w_1}$,
(v) $\Theta_{x,x,w}\Theta_{z,z,w} = 0$ *if and only if* $\langle x, z \mid w\rangle = 0$.

Proof. $(i), (ii), (iv)$ are trivial. For (iii) we have

$$\begin{aligned}\langle \Theta_{x,y,w}(y), y, \mid w\rangle &= \langle x\langle x, y \mid w\rangle, y \mid w\rangle\\ &= \langle y, x \mid w\rangle\langle x, y \mid w\rangle\\ &= |\langle x, y \mid w\rangle|^2 \geq 0.\end{aligned}$$

(v) If $\langle x, z \mid w\rangle = 0$ then, for all $y \in E$,

$$\Theta_{x,x,w}\Theta_{z,z,w}(y) = \Theta_{x,x,w}(z\langle z, y \mid w\rangle) = x\langle x, z \mid w\rangle\langle z, y \mid w\rangle = 0.$$

Conversely if $\Theta_{x,x,w}\Theta_{z,z,w} = 0$, then

$$\langle z, \Theta_{x,x,w}\Theta_{z,z,w}(x) \mid w\rangle = \langle z, x \mid w\rangle\langle x, z \mid w\rangle\langle z, x \mid w\rangle = 0$$

and hence we get

$$\begin{aligned}\|\langle x, z \mid w\rangle\|^4 &= \|\langle z, x \mid w\rangle\langle x, z \mid w\rangle\|^2\\ &= \|\langle z, x \mid w\rangle\langle x, z \mid w\rangle\langle z, x \mid w\rangle\langle x, z \mid w\rangle\| = 0,\end{aligned}$$

therefore $\langle x, z \mid w\rangle = 0$. □

From Theorem 2.1 we get the following corollary.

Corollary 2.2. *Let $\mathcal{A}$ be a C^*-algebra and E be an $\mathcal{A}$-2-inner product space. Let $x, z, w \in E$ be such that $\|x, w\| = \|z, w\| = 1$ and $\langle x, z \mid w\rangle = 0$. Then*

$$\Theta_{x,x,w} + \Theta_{z,z,w} \leq I,$$

where $I : E \to E$ is the identity operator.

Proof. Since $\|x, w\| = \|z, w\| = 1$, (2.1) implies that

$$|\langle z, y|w\rangle|^2 \leq |y, w|^2 - |\langle x, y|w\rangle|^2.$$

so

$$|\langle z, y|w\rangle|^2 + \langle x, y|w\rangle|^2 \leq |y, w|^2,$$

and this is equal to

$$\langle x\langle x, y|w\rangle, y \mid w\rangle + \langle z\langle z, y|w\rangle, y \mid w\rangle \leq \langle y, y \mid w\rangle.$$

Therefore, $\langle \Theta_{x,x,w}(y), y \mid w\rangle + \langle \Theta_{z,z,w}(y), y \mid w\rangle \leq \langle y, y \mid w\rangle$. □

3. Dragomir's inequality in an $\mathcal{A}$-2-inner product space

Let $\mathcal{A}$ be a C^*-algebra with unit e and E be an $\mathcal{A}$-2-inner product space. The following results may be stated, which are generalizations of Drogomir's results [3].

Theorem 3.1. *Let $\mathcal{A}$ be a C^*-algebra and E be an $\mathcal{A}$-2-inner product space. For $x, y, z, w \in E$ so that $\langle x, x \mid w\rangle$ is invertible in $\mathcal{A}$ and for each invertible $a \in \mathcal{A}$, we have*

$$\begin{aligned} &\|\langle y, x \mid w\rangle\langle x, x \mid w\rangle^{-1}\langle x, z \mid w\rangle - a^{-1}\langle y, z \mid w\rangle\| \\ &\leq \frac{\|z, w\|}{\|a\|}\|(a - e)\langle y, x \mid w\rangle\langle x, x \mid w\rangle^{-1}\langle x, y \mid w\rangle(a^* - e) \\ &+ \langle y, y \mid w\rangle - \langle y, x \mid w\rangle\langle x, x \mid w\rangle^{-1}\langle x, y \mid w\rangle\|^{\frac{1}{2}} \end{aligned} \tag{3.1}$$

and equality holds if there exists $a, b \in \mathcal{A}$, such that

$$x\langle x, x \mid w\rangle^{-1}\langle x, y \mid w\rangle a^* = y + zc + wba^*.$$

Proof.

$$\begin{aligned} &\left\langle x\langle x, x \mid w\rangle^{-1}\langle x, y \mid w\rangle - y(a^{-1})^*, z \mid w\right\rangle \\ &\times \left\langle z, x\langle x, x \mid w\rangle^{-1}\langle x, y \mid w\rangle - y(a^{-1})^* \mid w\right\rangle \\ &\leq \|z, w\|^2 \\ &\times \left\langle x\langle x, x \mid w\rangle^{-1}\langle x, y \mid w\rangle - y(a^{-1})^*, x\langle x, x \mid w\rangle^{-1}\langle x, y \mid w\rangle - y(a^{-1})^* \mid w\right\rangle. \end{aligned} \tag{3.2}$$

Since

$$
\begin{aligned}
&\langle x\langle x,x \mid w\rangle^{-1}\langle x,y \mid w\rangle - y(a^{-1})^*, x\langle x,x \mid w\rangle^{-1}\langle x,y \mid w\rangle - y(a^{-1})^* \mid w\rangle \\
&= \langle y,x \mid w\rangle\langle x,x \mid w\rangle^{-1}\langle x,x \mid w\rangle\langle x,x \mid w\rangle^{-1}\langle x,y \mid w\rangle - \langle y,x \mid w\rangle\langle x,x \mid w\rangle^{-1} \\
&\times \langle x,y \mid w\rangle(a^{-1})^* - a^{-1}\langle y,x \mid w\rangle\langle x,x \mid w\rangle^{-1}\langle x,y \mid w - a^{-1}\langle y,y \mid w\rangle(a^{-1})^* \\
&= a^{-1}(a\langle y,x \mid w\rangle\langle x,x \mid w\rangle^{-1}\langle x,y \mid w\rangle(a^*) - a\langle y,x \mid w\rangle\langle x,x \mid w\rangle^{-1}\langle x,y \mid w\rangle \\
&- \langle y,x \mid w\rangle\langle x,x \mid w\rangle^{-1}\langle x,y \mid w\rangle(a^*) + \langle y,y \mid w\rangle)(a^{-1})^* \\
&= a^{-1}((a-e)\langle y,x \mid w\rangle\langle x,x \mid w\rangle^{-1}\langle x,y \mid w\rangle(a^*-e) \\
&- \langle y,x \mid w\rangle\langle x,x \mid w\rangle^{-1}\langle x,y \mid w\rangle + \langle y,y \mid w\rangle)(a^{-1})^*
\end{aligned}
$$

and since

$$
\begin{aligned}
&\left\langle x\langle x,x \mid w\rangle^{-1}\langle x,y \mid w\rangle - y(a^{-1})^*, z \mid w\right\rangle \left\langle z, x\langle x,x \mid w\rangle^{-1}\langle x,y \mid w\rangle - y(a^{-1})^* \mid w\right\rangle \\
&= \left[\langle y,x \mid w\rangle\langle x,x \mid w\rangle^{-1}\langle x,z \mid w\rangle - a^{-1}\langle y,z \mid w\rangle\right] \\
&\times \left[\langle z,x \mid w\rangle\langle x,x \mid w\rangle^{-1}\langle x,y \mid w\rangle - \langle z,y \mid w\rangle(a^{-1})^*\right] \\
&= \left[\langle y,x \mid w\rangle\langle x,x \mid w\rangle^{-1}\langle x,z \mid w\rangle - a^{-1}\langle y,z \mid w\rangle\right] \\
&\times \left[\langle y,x \mid w\rangle\langle x,x \mid w\rangle^{-1}\langle x,z \mid w\rangle - a^{-1}\langle y,z \mid w\rangle\right]^*
\end{aligned}
$$

thus

$$
\begin{aligned}
&\|\langle y,x \mid w\rangle\langle x,x \mid w\rangle^{-1}\langle x,z \mid w\rangle - a^{-1}\langle y,z \mid w\rangle\|^2 \\
&= \| \left\langle x\langle x,x \mid w\rangle^{-1}\langle x,y \mid w\rangle - y(a^{-1})^*, z \mid w\right\rangle \\
&\times \left\langle z, x\langle x,x \mid w\rangle^{-1}\langle x,y \mid w\rangle - y(a^{-1})^* \mid w\right\rangle \| \\
&\leq \frac{\|z,w\|^2}{\|a\|^2}\|(a-e)\langle y,x \mid w\rangle\langle x,x \mid w\rangle^{-1}\langle x,y \mid w\rangle(a^*-e) \\
&+ \langle y,y \mid w\rangle - \langle y,x \mid w\rangle\langle x,x \mid w\rangle^{-1}\langle x,y \mid w\rangle\|
\end{aligned}
$$

From Lemma 2.1, the equality holds in (3.1), if there exists $b \in \mathcal{A}$ such that

$$
\begin{aligned}
&x\langle x,x \mid w\rangle^{-1}\langle x,y \mid w\rangle - y(a^{-1})^* \\
&= \frac{z}{\|z,w\|^2}\langle z, x\langle x,x \mid w\rangle^{-1}\langle x,y \mid w\rangle - y(a^{-1})^* \mid w\rangle + wb.
\end{aligned}
$$

Take $c = \frac{1}{\|z,w\|^2}\langle y,x \mid w\rangle\langle x,x \mid w\rangle^{-1}\langle z,x \mid w\rangle a^*$, then we have

$$x\langle x,x \mid w\rangle^{-1}\langle x,y \mid w\rangle a^* = y + zc + wba^*.$$

□

Putting $a = 2e$ in Theorem 2.1, we get the following result.

Corollary 3.1. *Let $\mathcal{A}$ be a C^*-algebra and E be an $\mathcal{A}$-2-inner product space. For $x,y,z,w \in E$ so that $\langle x,x \mid w\rangle$ is invertible in $\mathcal{A}$, then we have*

$$\|\langle y,x \mid w\rangle\langle x,x \mid w\rangle^{-1}\langle x,z \mid w\rangle\| \leq \frac{1}{2}(\|y,w\|\|z,w\| + \|\langle y,z \mid w\rangle\|).$$

Theorem 3.2. *Let $\mathcal{A}$ be a C^*-algebra and E be an $\mathcal{A}$-2-inner product space. If $x, y, z, v, w \in E$ are so that $\langle x, x \mid w\rangle$ is invertible in $\mathcal{A}$ and $\langle x, v \mid w\rangle = 0$ and $\|\langle v, v \mid w\rangle\| = 1$, then*

$$\begin{aligned}
&\|\langle y, x \mid w\rangle\langle x, x \mid w\rangle^{-1}\langle x, z \mid w\rangle\| \\
&\leq \frac{1}{2}(\|\langle y, y \mid w\rangle - \langle y, v \mid w\rangle\langle v, y \mid w\rangle\|^{1/2}\|\langle z, z \mid w\rangle \\
&- \langle z, v \mid w\rangle\langle v, z \mid w\rangle\|^{1/2} + \|\langle y, z \mid w\rangle - \langle y, v \mid w\rangle\langle v, z \mid w\rangle\|).
\end{aligned}$$

Proof. Take $v_1 = y - v\langle v, y \mid w\rangle$ and $v_2 = z - v\langle v, z \mid w\rangle$, then

$$\begin{aligned}
&\langle v_1, x \mid w\rangle\langle x, x \mid w\rangle^{-1}\langle x, v_2 \mid w\rangle \\
&= \langle y - v\langle v, y \mid w\rangle, x \mid w\rangle\langle x, x \mid w\rangle^{-1}\langle x, z - v\langle v, z \mid w\rangle \mid w\rangle \\
&= \langle y, x \mid w\rangle\langle x, x \mid w\rangle^{-1}\langle x, z \mid w\rangle
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
\langle v_1, v_2 \mid w\rangle &= \langle y - v\langle v, y \mid w\rangle, z - v\langle v, z \mid w\rangle \mid w\rangle \\
&= \langle y, z \mid w\rangle - \langle y, v \mid w\rangle\langle v, z \mid w\rangle - \langle y, v \mid w\rangle\langle v, z \mid w\rangle \\
&+ \langle y, v \mid w\rangle\langle v, v \mid w\rangle\langle v, z \mid w\rangle \\
&\leq \langle y, z \mid w\rangle - \langle y, v \mid w\rangle\langle v, z \mid w\rangle
\end{aligned} \tag{3.4}$$

and

$$\begin{aligned}
\langle v_1, v_1 \mid w\rangle &= \langle y - v\langle v, y \mid w\rangle, y - v\langle v, y \mid w\rangle \mid w\rangle \\
&= \langle y, y \mid w\rangle - \langle y, v \mid w\rangle\langle v, y \mid w\rangle - \langle y, v \mid w\rangle\langle v, y \mid w\rangle \\
&+ \langle y, v \mid w\rangle\langle v, v \mid w\rangle\langle v, y \mid w\rangle \\
&\leq \langle y, y \mid w\rangle - \langle y, v \mid w\rangle\langle v, y \mid w\rangle
\end{aligned} \tag{3.5}$$

and

$$\begin{aligned}
\langle v_2, v_2 \mid w\rangle &= \langle z - v\langle v, z \mid w\rangle, z - v\langle v, z \mid w\rangle \mid w\rangle \\
&= \langle z, z \mid w\rangle - \langle z, v \mid w\rangle\langle v, z \mid w\rangle - \langle z, v \mid w\rangle\langle v, z \mid w\rangle \\
&+ \langle z, v \mid w\rangle\langle v, v \mid w\rangle\langle v, z \mid w\rangle \\
&\leq \langle z, z \mid w\rangle - \langle z, v \mid w\rangle\langle v, z \mid w\rangle
\end{aligned} \tag{3.6}$$

From Corollary 3.1, the relations (3.3), (3.4), (3.5) and (3.6), we get

$$\begin{aligned}
&\|\langle y, x \mid w\rangle\langle x, x \mid w\rangle^{-1}\langle x, z \mid w\rangle\| \\
&= \|\langle v_1, x \mid w\rangle\langle x, x \mid w\rangle^{-1}\langle x, v_2 \mid w\rangle\| \\
&\leq \frac{1}{2}(\|v_1, w\|\|v_2, w\| + \|\langle v_1, v_2 \mid w\rangle\|) \\
&\leq \frac{1}{2}(\|\langle y, y \mid w\rangle - \langle y, v \mid w\rangle\langle v, y \mid w\rangle\|^{1/2}\|\langle z, z \mid w\rangle \\
&- \langle z, v \mid w\rangle\langle v, z \mid w\rangle\|^{1/2} + \|\langle y, z \mid w\rangle - \langle y, v \mid w\rangle\langle v, z \mid w\rangle\|).
\end{aligned}$$

□

4. Applications

Let (Ω, Σ, μ) be a measure space consisting of a set Ω, a σ-algebra Σ of subsets of Ω and a positive measure μ on Σ with values in $\mathbb{R}\cup\{\infty\}$. Denote by $L_\upsilon^2(\Omega)$ the Hilbert space of all real-valued functions α defined on Ω that are 2-υ-integrable on Ω, i.e.

$$\int_\Omega \upsilon(s)|\alpha(s)|^2 \mathrm{d}\mu(s) < \infty,$$

where $\upsilon : \Omega \to [0, \infty)$ is a measurable fuction on Ω. In the following, we define a 2-inner product and a 2-norm on $L_\upsilon^2(\Omega)$ by

$$\langle \alpha, \beta \mid \gamma\rangle_\upsilon := 1/2 \int_\Omega \int_\Omega \upsilon(s)\upsilon(t) \begin{vmatrix} \beta(s) & \beta(t) \\ \gamma(s) & \gamma(t) \end{vmatrix} \begin{vmatrix} \alpha(s) & \alpha(t) \\ \gamma(s) & \gamma(t) \end{vmatrix} \mathrm{d}\mu(s)\mathrm{d}\mu(t),$$

and

$$\|\alpha, \gamma\|_\upsilon := \left(1/2 \int_\Omega \int_\Omega \upsilon(s)\upsilon(t) \begin{vmatrix} \alpha(s) & \alpha(t) \\ \gamma(s) & \gamma(t) \end{vmatrix}^2 \mathrm{d}\mu(s)\mathrm{d}\mu(t)\right)^{1/2}.$$

A simple calculation with integrals reveals that

$$\langle \alpha, \beta \mid \gamma\rangle_\upsilon := \begin{vmatrix} \int_\Omega \upsilon\alpha\beta \mathrm{d}\mu & \int_\Omega \upsilon\alpha\gamma \mathrm{d}\mu \\ \int_\Omega \upsilon\beta\gamma \mathrm{d}\mu & \int_\Omega \upsilon\gamma^2 \mathrm{d}\mu \end{vmatrix} \tag{4.1}$$

and

$$\|\alpha, \gamma\|_\upsilon := \begin{vmatrix} \int_\Omega \upsilon\alpha^2 \mathrm{d}\mu & \int_\Omega \upsilon\alpha\gamma \mathrm{d}\mu \\ \int_\Omega \upsilon\alpha\gamma \mathrm{d}\mu & \int_\Omega \upsilon\gamma^2 \mathrm{d}\mu \end{vmatrix}^{1/2} \tag{4.2}$$

where, for simplicity, instead of $\int_\Omega \upsilon(s)\alpha(s)\beta(s)\mathrm{d}\mu(s)$, we have written $\int_\Omega \upsilon\alpha\beta\mathrm{d}\mu$. From Theorem 2.1, we have the following interesting determinantal integral inequality.

Proposition 4.1. *Let $\mathcal{A} = \mathbb{C}$ and $L_\upsilon^2(\Omega)$ be an $\mathcal{A}$-2-inner product space and let $\alpha, \beta, \eta, \gamma \in L_\upsilon^2(\Omega)$, $\alpha \neq 0, \beta \neq 0$ be such that $\langle \alpha, \eta \mid \gamma\rangle = 0$, then*

$$\left(\begin{vmatrix} \int_\Omega \upsilon\eta\beta \mathrm{d}\mu & \int_\Omega \upsilon\eta\gamma \mathrm{d}\mu \\ \int_\Omega \upsilon\beta\gamma \mathrm{d}\mu & \int_\Omega \upsilon\gamma^2 \mathrm{d}\mu \end{vmatrix}\right)^2 \leq \frac{\begin{vmatrix} \int_\Omega \upsilon\eta^2 \mathrm{d}\mu & \int_\Omega \upsilon\eta\gamma \mathrm{d}\mu \\ \int_\Omega \upsilon\eta\gamma \mathrm{d}\mu & \int_\Omega \upsilon\gamma^2 \mathrm{d}\mu \end{vmatrix}}{\begin{vmatrix} \int_\Omega \upsilon\alpha^2 \mathrm{d}\mu & \int_\Omega \upsilon\alpha\gamma \mathrm{d}\mu \\ \int_\Omega \upsilon\alpha\gamma \mathrm{d}\mu & \int_\Omega \upsilon\gamma^2 \mathrm{d}\mu \end{vmatrix}} \times$$

$$\left(\begin{vmatrix} \int_\Omega \upsilon\alpha^2 \mathrm{d}\mu & \int_\Omega \upsilon\alpha\gamma \mathrm{d}\mu \\ \int_\Omega \upsilon\alpha\gamma \mathrm{d}\mu & \int_\Omega \upsilon\gamma^2 \mathrm{d}\mu \end{vmatrix} \begin{vmatrix} \int_\Omega \upsilon\beta^2 \mathrm{d}\mu & \int_\Omega \upsilon\beta\gamma \mathrm{d}\mu \\ \int_\Omega \upsilon\beta\gamma \mathrm{d}\mu & \int_\Omega \upsilon\gamma^2 \mathrm{d}\mu \end{vmatrix} - \begin{vmatrix} \int_\Omega \upsilon\alpha\beta \mathrm{d}\mu & \int_\Omega \upsilon\alpha\gamma \mathrm{d}\mu \\ \int_\Omega \upsilon\beta\gamma \mathrm{d}\mu & \int_\Omega \upsilon\gamma^2 \mathrm{d}\mu \end{vmatrix}^2\right).$$

The equality holds if and only if

$$\beta - \frac{\alpha \begin{vmatrix} \int_\Omega \upsilon\alpha\beta \mathrm{d}\mu & \int_\Omega \upsilon\alpha\gamma \mathrm{d}\mu \\ \int_\Omega \upsilon\beta\gamma \mathrm{d}\mu & \int_\Omega \upsilon\gamma^2 \mathrm{d}\mu \end{vmatrix}}{\begin{vmatrix} \int_\Omega \upsilon\alpha^2 \mathrm{d}\mu & \int_\Omega \upsilon\alpha\gamma \mathrm{d}\mu \\ \int_\Omega \upsilon\alpha\gamma \mathrm{d}\mu & \int_\Omega \upsilon\gamma^2 \mathrm{d}\mu \end{vmatrix}} = \frac{\eta \begin{vmatrix} \int_\Omega \upsilon\eta\beta \mathrm{d}\mu & \int_\Omega \upsilon\eta\gamma \mathrm{d}\mu \\ \int_\Omega \upsilon\beta\gamma \mathrm{d}\mu & \int_\Omega \upsilon\gamma^2 \mathrm{d}\mu \end{vmatrix}}{\begin{vmatrix} \int_\Omega \upsilon\eta^2 \mathrm{d}\mu & \int_\Omega \upsilon\eta\gamma \mathrm{d}\mu \\ \int_\Omega \upsilon\eta\gamma \mathrm{d}\mu & \int_\Omega \upsilon\gamma^2 \mathrm{d}\mu \end{vmatrix}} + \gamma a \quad (a \in \mathbb{C}).$$

References

[1] Y.J. Cho, P.C.S. Lin, S.S. Kim and A. Misiak, *Theory of 2-Inner Product Spaces*, Nova Science Publishers, New York, 2001.

[2] S.S. Dragomir, A.C. Gosa, A generalisation of an Ostrowski inequality in inner product spaces, *Inequality theory and applications. Vol. 4*, 61–64, Nova Science Publishers, New York, 2007.

[3] S.S. Dragomir, Refinement of Buzano's and Kurepa's inequalities in inner product spaces, *Facta universitatis (NIS). Ser. Math. Inform.* **20** (2005), 65–73.

[4] L. Arambašić and R. Rajić, Ostrowski's inequality in pre-Hilbert C^*-modules, *Math. Inequal. Appl.* **12** (2009), no. 1, 217–226.

[5] T. Mehdiabad Mahchari and A. Nazari, 2-Hilbert C^*-modules and some Gruss type inequalities in $\mathcal{A}$-2-inner product space, *Math. Inequal. Appl.* **18** (2015), no. 2, 721–754.

[6] B. Mohebbi Najmabadi and T.L. Shateri, 2-inner product which takes values on a locally C^*-algebra, *Indian J. Math. Soc.* **85** (2018), no. 1-2, 218–226.

[7] A.M. Ostrowski, *Vorlesungen über Differential und Integralrechnung II*, Birkhaüser, Basel, 1951.

[8] C.E.M. Pearce, J. Pečarić and S. Varošanec, An integral analogue of the Ostrowski inequality, *J. Inequal. Appl.* **2** (1998), no. 3, 275–283.

[9] H. Šikić and T. Šikić, A note on Ostrowski's inequality, *Math. Ineq. Appl.* **4** (2001), no.2, 297–299.

Behrooz Mohebbi Najmabadi

Department of Mathematics and Computer Sciences, Hakim Sabzevari University, Sabzevar, P.O. Box 397, Iran.

E-mail address: `behrozmohebbi1351@gmail.com`

Tayebe Lal Shateri

Department of Mathematics and Computer Sciences, Hakim Sabzevari University, Sabzevar, P.O. Box 397, Iran.

E-mail address: `t.shateri@hsu.ac.ir`

Received: July 23, 2018; Revised: January 20, 2019; Accepted: February 7, 2019